Fragmentation under the scaling symmetry and turbulent cascade with intermittency

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1. Motivation and objectives

Fragmentation plays an important role in a variety of physical, chemical, and geological processes. Examples include atomization in sprays, crushing of rocks, explosion and impact of solids, polymer degradation, etc. Although each individual action of fragmentation is a complex process, the number of these elementary actions is large. It is natural to abstract a simple 'effective' scenario of fragmentation and to represent its essential features. One of the models is the fragmentation under the scaling symmetry: each breakup action reduces the typical length of fragments, $r \Rightarrow \alpha r$, by an independent random multiplier $\alpha \, (0 < \alpha < 1)$, which is governed by the fragmentation intensity spectrum $q(\alpha)$, $\int_0^1 q(\alpha)d\alpha = 1$. This scenario has been proposed by Kolmogorov (1941), when he considered the breakup of solid carbon particle. Describing the breakup as a random discrete process, Kolmogorov stated that at latest times, such a process leads to the log-normal distribution. In Gorokhovski & Saveliev (2003), the fragmentation under the scaling symmetry has been reviewed as a continuous evolution process with new features established.

The objective of this paper is twofold. First, the paper synthesizes and completes theoretical part of Gorokhovski & Saveliev (2003). Second, the paper shows a new application of the fragmentation theory under the scale invariance. This application concerns the turbulent cascade with intermittency. We formulate here a model describing the evolution of the velocity increment distribution along the progressively decreasing length scale. The model shows that when the turbulent length scale gets smaller, the velocity increment distribution has central growing peak and develops stretched tails. The intermittency in turbulence is manifested in the same way: large fluctuations of velocity provoke highest strain in narrow (dissipative) regions of flow.

2. Universalities of fragmentation under the scaling symmetry

2.1. The evolution equation for normalized distribution of fragments and its steady-state solution

The population balance in the case of fragmentation under the scaling symmetry evolves according to the following integro-differential equation (see, for example, Gorokhovski & Saveliev (2003)):

$$\frac{\partial f}{\partial t} = (\widehat{I}_+ - 1)\nu f \qquad (2.1)$$

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where $f(r,t)$ is the normalized distribution of size, ν is constant breakup frequency, $\int_0^\infty f(r)dr = 1$ and

$$\widehat{I}_+ f = \int_0^1 f\left(\frac{r}{\alpha}\right) q(\alpha) \frac{d\alpha}{\alpha} \tag{2.2}$$

is the operator of fragmentation. To fulfill the evolution of distribution with time, we consider $q(\alpha)$ to be different from delta function. The ultimate steady-state solution of equation (2.1) is delta function:

$$f(r) = \delta(r) \tag{2.3}$$

To get (2.3), remark that $\widehat{I}_+ \delta(r) = \int_0^1 \frac{d\alpha}{\alpha} q(\alpha) \delta\left(\frac{r}{\alpha}\right) = \delta(r)$, and then from $\left(\widehat{I}_+ - 1\right) f = 0$, it follows that $\widehat{I}_+ \delta(r) - \delta(r) = 0$. The question is: How does the distribution $f(r,t)$ evolve to the ultimate steady-state solution (2.3)? This question can not be completely answered since the solution of the evolution equation (2.1) requires knowledge of the spectrum $q(\alpha)$, which is principally unknown function. At the same time, the operator $\widehat{I}_+$ in equation (2.1) is invariant under the group of scaling transformations $(r \to \alpha r)$. Due to this symmetry, the evolution of the distribution $f(r,t)$ to the ultimate steady-state solution (2.3) goes at least, through two intermediate asymptotics. Evaluating these intermediate asymptotics does not require knowledge of entire function $q(\alpha)$ - only its first two logarithmic moments, and further only the ratio of these moments in the long-time limit, determine the behavior of the solution to equation (2.1). These two universalities are shown as follows.

2.2. *First and second universalities*

The asymptotic solution of (2.1) is (Gorokhovski & Saveliev 2003):

$$f(r, t \to \infty) = \frac{1}{R} \frac{1}{\sqrt{2\pi \langle \ln^2 \alpha \rangle \nu t}} \exp\left(-\frac{\langle \ln \alpha \rangle^2}{2 \langle \ln^2 \alpha \rangle} \nu t\right) \times$$
$$\times \exp\left(-\frac{\langle \ln(r/R) \rangle^2}{2 \langle \ln^2 \alpha \rangle \nu t}\right) \left(\frac{R}{r}\right)^{1 - \langle \ln \alpha \rangle / \langle \ln^2 \alpha \rangle} \tag{2.4}$$

where R denotes the initial length scale. The expression (2.4) confirms the main result of Kolmogorov (1941): the long-time limit distribution is log-normal (*first universality* with two parameters, which are the first and the second logarithmic moments of the fragmentation intensity spectrum). At the same time, it is seen from (2.4), that the second multiplier tends to unity as time progresses. This implies that *only one universal parameter*, $\frac{\langle \ln \alpha \rangle}{\langle \ln^2 \alpha \rangle}$, controls the last stage of the fragmentation process (*second universality*):

$$f(r,t) \propto \left(\frac{1}{r}\right)^{1 - \frac{\langle \ln \alpha \rangle}{\langle \ln^2 \alpha \rangle}} \tag{2.5}$$

2.3. *First universality and Fokker-Planck equation*

The emerging corollary from the first universality is as follows: changing of higher moments $\langle \ln^k \alpha \rangle$, $k > 2$ in equation (2.1) does not affect its solution at times sufficiently larger than the life time of the breaking fragment. Then the moments $\langle \ln^k \alpha \rangle$, $k > 2$ can be simply equated to zero (rather than making the broadly-used assumption about the smallness of latest). Consequently, by expanding $\frac{1}{\alpha} f\left(\frac{r}{\alpha}\right)$ on powers of $\ln \alpha$ in (2.1), and
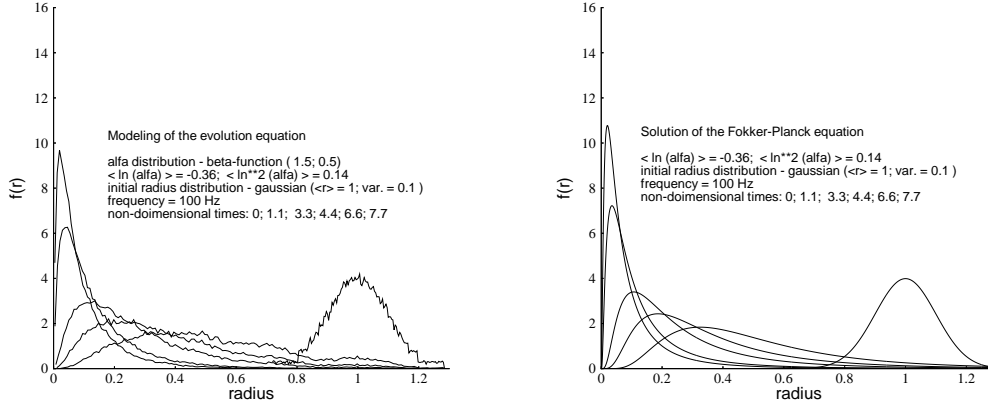
FIGURE 1. Size distributions computed by direct modeling of the evolution equation with presumed Gaussian spectrum of breakup intensity, $q(\alpha)$, (on the left-side) and by analitical solution of Fokker-Planck equation with $\langle \ln \alpha \rangle$ and $\langle \ln^2 \alpha \rangle$ taken from presumed $q(\alpha)$.

by setting to zero the third and all higher logarithmic moments, the evolution equation (2.1) reduces *exactly* to the Fokker-Planck equation (Gorokhovski & Saveliev 2003):

$$\frac{\partial f(r)}{\partial t} = \left[-\frac{\partial}{\partial r} r \langle \ln \alpha \rangle + \frac{1}{2!} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} r \langle \ln^2 \alpha \rangle \right] \nu f(r) \qquad (2.6)$$

or in terms of logarithm of size distribution $\Phi(x = \ln r)$, one yields:

$$\frac{\partial \Phi(x)}{\partial t} = \left[-\frac{\partial}{\partial x} \langle \ln \alpha \rangle + \frac{1}{2!} \frac{\partial^2}{\partial x^2} \langle \ln^2 \alpha \rangle \right] \nu \Phi(x) \qquad (2.7)$$

The solution of (2.6) verifies to be:

$$f(r,t) = \frac{1}{r} \int_0^\infty \frac{1}{\sqrt{2\pi \langle \ln^2 \alpha \rangle \nu t}} \exp \left[-\frac{\left(\ln \frac{r_0}{r} + \langle \ln \alpha \rangle \nu t \right)^2}{2 \langle \ln^2 \alpha \rangle \nu t} \right] f_0(r_0) dr_0 \qquad (2.8)$$

where $f_0(r_0)$ is the initial distribution.

If $q(\alpha)$ is presumed, we can compare the Monte-Carlo simulation of the evolution equation (2.1) with analytical solution (2.8), where $\langle \ln \alpha \rangle$ and $\langle \ln^2 \alpha \rangle$ are calculated from the presumed $q(\alpha)$. In Fig.1, on the left hand-side, we show the distribution from Monte Carlo computation of (2.1) at different non-dimensional time νt, with $q(\alpha)$ presumed as Gaussian; $\langle \ln \alpha \rangle = -0.36$ and $\langle \ln^2 \alpha \rangle = 0.14$. The same moments have been prescribed in (2.8) to compute the evolution of distribution by Fokker-Planck equation (right hand-side of Fig.1). It is seen that at $\nu t = 1.1$ and later, the Monte Carlo simulation of (2.1) and analytical solution of Fokker-Planck equation (2.8) match each other. We performed such a comparison using different shapes of $q(\alpha)$ and the emerging picture shows that after a certain time (the weaker the spectrum of breakup intensity is, the larger this time is), the Monte Carlo solution of the evolution equation and the analytical solution of Fokker-Planck equation were practically the same.

2.4. *Second universality, fractals and Boltzman distribution. Identification of* $\frac{\langle \ln^2 \alpha \rangle}{\langle \ln \alpha \rangle}$

The power distribution (2.5) implies the fractal properties of formed fragments in the long-time limit. The dimension of such a fractal object is defined by the ratio $\frac{\langle \ln \alpha \rangle}{\langle \ln^2 \alpha \rangle}$. Setting in (2.5) $x = \ln r$, one yields:

$$\Phi(x) = r \cdot f(r,t) \; \propto \; e^{-\frac{x}{h}} \tag{2.9}$$

where

$$h = -\frac{\langle \ln^2 \alpha \rangle}{\langle \ln \alpha \rangle} \tag{2.10}$$

From (2.9), one can see that in the fragmentation process, the asymptotic power distribution (2.5) plays the same role as the Boltzmann distribution in problems of statistical physics. This gives an idea for the choice of $\frac{\langle \ln \alpha \rangle}{\langle \ln^2 \alpha \rangle}$ by making use of the theory of Einstein on the Brownian motion. In this theory, the coefficient of diffusion in the Fokker-Planck equation is represented by the product *mobility×energy*, while the drift velocity is given by the product *mobility×force*. The ratio characterizes the typical length scale. In this spirit, making analogy with (2.7), one gives for the normalized typical length scale r_*:

$$\frac{\langle \ln^2 \alpha \rangle}{\langle \ln \alpha \rangle} = \ln \left(\frac{r_*}{r_0} \right) \tag{2.11}$$

This scale may characterize the dominant mechanism of the cascade fragmentation (an example of r_* can be found in Gorokhovski (2001)).

3. Application to the turbulent cascade with intermittency

The cascade in isotropic turbulence with intermittency in the velocity field may also be viewed in the framework of fragmentation under the scaling symmetry. Here, the energy of larger unstable eddies is transferred to smaller one at a fluctuating rate. It is clear that controlling of each elementary breakup of eddy is useless and impossible task, since the number of degrees of freedom to produce each turbulent structure is very large. The very simple way is again, to assume that at each repetitive step of cascade, the probability to find the velocity scale of a 'daughter' eddy is independent of the velocity scale of its 'mother' eddy; *i.e.* when the turbulent length scale r gets smaller, the velocity increment, $\triangle_r \mathbf{v}(x) = |\mathbf{v}(x+r) - \mathbf{v}(x)|$, is changed by independent positive random multiplier:

$$\triangle_r \mathbf{v} = \alpha \triangle_l \mathbf{v}, \text{ with } r \leq l \tag{3.1}$$

The formulation (3.1) is similar to Castaign *et al.* (1990), Castaign *et al.* (1993), Castaign (1996), Kahalerras *et al.* (1997), Naert *et al.* (1998). In these papers, the measurements of statistics of the velocity increment at the progressively reduced length scale showed distributions with stretched tails and sharp central peak, *i.e.* at small length scales, the small amplitude events alternate with events of strong velocity variation. This provokes effect of intermittency: highest strain in narrow regions of flow. By Taylor's hypothesis of 'frozen turbulence', it is traditional to evaluate the velocity increment by one-point measurement of the velocity time increment. Consequently, the penetration towards smaller scale in the turbulent cascade may be viewed as evolution 'in time' by: $\tau_i = \ln(L_{int}/l_i)$ (Friedrich & Peinke 1997), where L_{int} is integral length scale and l_i is the eddy scale. Then we may compare equation (2.8) with measured distribution of the velocity increment for different length scales. Another way, is to compare equations (2.8) and (2.11)
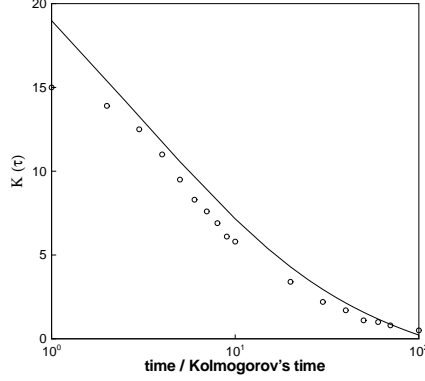
FIGURE 2. Evolution of the flatness factor $K(\tau) = \left\langle (\triangle_\tau \mathbf{v})^4 \right\rangle / \left\langle (\triangle_\tau \mathbf{v})^2 \right\rangle^2 - 3$ for the PDF of the time velocity increment (continuous line: model; symbol: Mordant *et al.* (2001)).

directly with Lagrangian velocity statistics measured by Mordant *et al.* (2001) in fully developed turbulence. Assuming that at integral time scale, the distribution of the velocity increment is Gaussian (see, for example, Obukhov's (1959) Lagrangian theory of isotropic turbulence), the expression (2.8) can be written as:

$$f(\triangle_\tau \mathbf{v}, \tau_*) = \frac{1}{\triangle_\tau \mathbf{v}} \times$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi \left\langle \ln^2 \alpha \right\rangle \tau_*}} \exp\left[-\frac{\left(\ln \frac{\triangle \mathbf{V}}{\triangle_\tau \mathbf{v}} + \left\langle \ln \alpha \right\rangle \tau_* \right)^2}{2 \left\langle \ln^2 \alpha \right\rangle \tau_*} \right] Gauss(\triangle \mathbf{V}) d(\triangle \mathbf{V}) \quad (3.2)$$

where τ_* is assumed here to be $\ln(T_{int}/\tau_i)$ (T_{int} is integral time scale and τ_i is eddy turbulent time scale). The crucial problem in (3.2) concerns definition of $\left\langle \ln^2 \alpha \right\rangle$ and $\left\langle \ln \alpha \right\rangle$. It has been recognized in two recent papers of Lundgren (2002) and Gange *et al.* (2003), that the typical turbulent length scale, at which Kolmogorov's scaling takes place at finite large Reynolds number, is close to Taylor micro-scale, λ. In this spirit, the expression (2.11) is formulated here as: $\left\langle \ln^2 \alpha \right\rangle / \left\langle \ln \alpha \right\rangle = \ln(\lambda/L_{int})$ and further $\left\langle \ln \alpha \right\rangle = const \cdot \ln(\lambda/L_{int})$. In Fig. 3, the computed PDF's of the velocity increment are shown against measurements of Mordant *et al.* (2001) for $Re_\lambda = 740$; $\mathbf{v}_{rms} = 0.98 \ m/s$; $T_{int} = 23 \ ms$; $\tau_\eta = 0.2 \ ms$ (variations have been normalized to unit variance in order to emphasize changes in the functional form). Without special fit to each experimental profile (computation requires only one constant giving a good fit to the experimental evolution of the flatness parameter, Fig. 2), it is seen that expression (3.2) is in agreement with the measured PDF. The computed PDF's reproduce the progressive non-Gaussianity with development of stretched tails as time increment becomes smaller.

4. Conclusion and future work

The main result in this paper concerns formulation of the turbulent cascade with intermittency in the framework of the fragmentation theory with scale invariance. The model of evolution of the velocity increment distribution along the progressively decreasing length scale is proposed. The Monte Carlo simulation of the evolution equation with
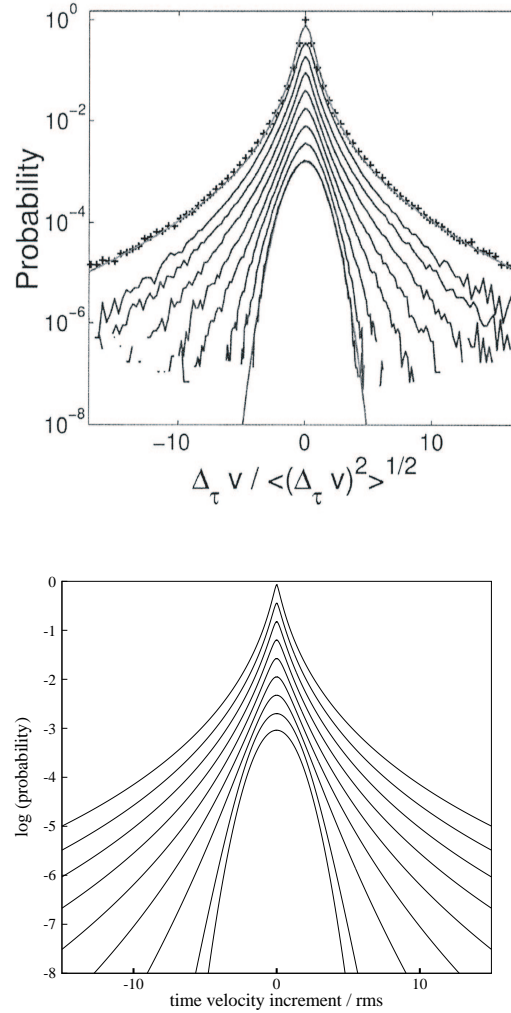
FIGURE 3. Experimental from Mordant *et al.* (2001) (upper part) and corresponding modeled (bottom part) PDF of the normalized increment $\triangle_\tau \mathbf{v} / \left\langle (\triangle_\tau \mathbf{v})^2 \right\rangle^{1/2}$ at $Re_\lambda = 740$. The curves are shifted for clarity. From top to bottom: $\tau = 0.15; 0.3; 0.6; 1.2; 2.5; 5; 10; 20; 23\ ms..$

presumed fragmentation spectrum showed that the solution matches the Fokker-Planck approximation. We showed that the long-time evolution towards fractals is similar to the Boltzmann distribution in the statistical physics. This allowed to represent the ratio of two first logarithmic moments of the fragmentation intensity spectrum as a typical scale, at which the cascade fragmentation is manifested. The distribution of velocity increment showed that when the turbulent length scale gets smaller, this distribution has central growing peak and develops stretched tails. This is similar to what we know on the intermittency in turbulence from papers of Castaign *et al.* (1990, 1993, 1996, 1997, 1998). In addition, the model proposed can be readily applied in LES computation of turbulent intermittent flow with dispersed particles. In such computation we often need to predict the flow properties at scales substantially smaller than the grid size. The knowledge of the

velocity increment at these scales may improve simulation of light particles dispersion, droplet evaporation involving blowing effects, and sub-grid vapor/air mixing.

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